Effects of Transverse Normal and Shear Strains in Orthotropic Shells

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Introduction

PLATE and shell structures in missile, space, and nuclear applications are often subjected to severe operational conditions. Classical methods of analysis based on the Kirchhoff-Love (K-L) hypothesis (implying neglect of transverse shear deformation and transverse normal stress) may not be applicable in such cases. This is especially true for certain composite and refractory materials which show a high degree of anisotropy in physical and mechanical properties.

Typical of such materials is pyrolytic graphite (PG), perhaps the best known of the many pyrolytic refractory materials. Noteworthy among its unusual properties[1] are the following: 1) PG is transversely isotropic. 2) The ratio of in-plane Young's modulus to transverse shear modulus (E/G_{13}) for PG varies between 20–50, as compared to E/G of 2.6 for an isotropic material with $\nu = 0.3$. 3) The ratio of transverse to in-plane thermal expansion coefficients (α_{33}/α_{11}) varies approximately between 10 and 30. 4) The in-plane Poisson's ratio for PG is negative ($\nu = -0.21$) while the transverse ratio ($\nu_{13} = 0.90$–1.0) is quite high.

The purpose of the present Note is to illustrate the significance of transverse shear, normal stress effects, and edge conditions on the stress and displacement computations for thin shells subjected to thermal loading and also to point out the hazards of employing classical shell theories for analysis of structures of pyrolytic graphite-type materials.

Governing Equations

The equations of motion for orthotropic, axisymmetric shells of revolution are derived in Ref. 2 from Hamilton's Principle and take into account the effects of transverse shear deformation, transverse normal stress (both elastic and thermal), rotatory inertia, and small finite deflections. For linear static problems, they reduce to the following.

Displacements

$$u_1 = u(\alpha_1, \alpha_2, 0) + \zeta\beta(\alpha_1, \alpha_2, 0)$$
$$u_3 = w(\alpha_1, \alpha_2, 0) + \zeta\lambda(\alpha_1, \alpha_2, 0) + \frac{\zeta^2}{z}\gamma(\alpha_1, \alpha_2, 0) \qquad (1)$$

Here, ζ is the coordinate in the thickness direction measured from the shell reference surface, u_1 and u_3 are displacement components in the axial and normal directions, respectively, and u, w, β, λ, γ are displacement functions to be determined to satisfy the shell equilibrium equations and boundary conditions. In classical shell theory, $\lambda = \gamma = 0$, u, and w represent displacements of the shell reference surface and β is the change in slope of the normal to the reference surface.

Strain-displacement relations

$$\begin{aligned} e_{11} &= (u_{1,1} + H_{1,3}u_3)/H_1 \\ e_{22} &= (H_{2,3}u_3 + H_{2,1}u_1/H_1)/H_2 \\ e_{33} &= u_{3,3} \\ 2e_{13} &= H_1(u_1/H_1)_{,3} + u_{3,1}/H_1 \end{aligned} \qquad (2)$$

where e_{ij} represent linear strain components, H_1, H_2 are Lamé constants, and a comma represents differentiation.

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Stress-strain relations

Hooke's Law for an orthotropic material may be written as[3]

$$\begin{Bmatrix} e_{11}-\alpha_{11}T \\ e_{22}-\alpha_{22}T \\ e_{33}-\alpha_{33}T \\ e_{13} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -1/E_{12} & -1/E_{13} & 0 \\ -1/E_{21} & 1/E_2 & -1/E_{23} & 0 \\ -1/E_{31} & -1/E_{32} & 1/E_3 & 0 \\ 0 & 0 & 0 & 1/2G_{13} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{13} \end{Bmatrix} \qquad (3)$$

where E_i is the elastic modulus in the i direction, G_{13} the transverse shear modulus, ν_{ij} is Poisson's ratio (defined as the negative of the ratio of strain in the j-direction to the strain in the i-direction due to a stress in the i-direction), and $E_{ij} = E_i/\nu_{ij} = E_j/\nu_{ji}$ $(i, j = 1, 2, 3)$, $T(\alpha_1, \alpha_2, \zeta)$, the temperature measured from the stress-free temperature of the material, and α_{ii} $(i = 1, 2, 3)$ are the linear thermal expansion coefficients.

In terms of stresses, Eq. (3) becomes

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} E_1^* & E_{12}^* & E_{13}^* & 0 \\ E_{21}^* & E_2^* & E_{23}^* & 0 \\ E_{31}^* & E_{32}^* & E_3^* & 0 \\ 0 & 0 & 0 & 2G_{13} \end{bmatrix} \begin{Bmatrix} e_{11}-\alpha_{11}T \\ e_{22}-\alpha_{22}T \\ e_{33}-\alpha_{33}T \\ e_{13} \end{Bmatrix} \qquad (4)$$

where

$$E_1^* = (1-\nu_{23}\nu_{32})VE_1 \quad E_{12}^* = (\nu_{21}+\nu_{23}\nu_{31})VE_1 = E_{21}^*$$
$$E_2^* = (1-\nu_{31}\nu_{13})VE_2 \quad E_{13}^* = (\nu_{13}+\nu_{21}\nu_{32})VE_1 = E_{31}^*$$
$$E_3^* = (1-\nu_{12}\nu_{21})VE_3 \quad E_{23}^* = (\nu_{32}+\nu_{12}\nu_{31})VE_2 = E_{32}^*$$

and

$$V = (1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{12}\nu_{23}\nu_{31})^{-1}$$

Stress resultants

Stress resultants are defined as

$$\begin{Bmatrix} N_\delta \\ M_\delta \\ P_\delta \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\delta\delta} \begin{Bmatrix} 1 \\ \zeta \\ \zeta^2/2 \end{Bmatrix} (1+\zeta R_\tau)\,d\zeta \quad (\delta \neq \tau;\ \delta, \tau = 1, 2)$$
$$\begin{Bmatrix} Q_1 \\ S_1 \\ T_1 \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{13} \begin{Bmatrix} 1 \\ \zeta \\ \zeta^2/2 \end{Bmatrix} (1+\zeta/R_1)\,d\zeta \qquad (5)$$
$$\begin{Bmatrix} A \\ B \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{33} \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} (1+\zeta/R_1)(1+\zeta/R_2)\,d\zeta$$
$$\begin{Bmatrix} N_\delta^T \\ M_\delta^T \\ P_\delta^T \end{Bmatrix} = \int_{-h/2}^{h/2} k_{\delta\delta} T \begin{Bmatrix} 1 \\ \zeta \\ \zeta^2/2 \end{Bmatrix} (1+\zeta/R_\tau)\,d\zeta \quad (\delta \neq \tau;\ \delta, \tau = 1, 2)$$
$$\begin{Bmatrix} A^T \\ B^T \end{Bmatrix} = \int_{-h/2}^{h/2} k_{33} T \begin{Bmatrix} 1 \\ \zeta \end{Bmatrix} (1+\zeta/R_1)(1+\zeta/R_2)\,d\zeta$$
$$k_{11} = E_1^*\alpha_{11} + E_{12}^*\alpha_{22} + E_{13}^*\alpha_{33}$$
$$k_{22} = E_{21}^*\alpha_{11} + E_2^*\alpha_{22} + E_{23}^*\alpha_{33}$$
$$k_{33} = E_{31}^*\alpha_{11} + E_{32}^*\alpha_{22} + E_3^*\alpha_{33}$$

Integrated equilibrium equations

$$\begin{aligned} (A_2N_1)_{,1} - A_{2,1}N_2 + A_1A_2Q_1/R_1 + q_1 &= 0 \\ (A_2Q_1)_{,1} - A_1A_2(N_1/R_1 + N_2/R_2 + q_3) &= 0 \\ (A_2M_1)_{,1} - A_{2,1}M_2 - A_1A_2(Q_1 - m_1) &= 0 \\ (A_2S_1)_{,1} - A_1A_2(M_1/R_1 + M_2/R_2 + A + m_3) &= 0 \\ (A_2T_1)_{,1} - A_1A_2\left(P_1/R_1 + P_2/R_2 + B + \frac{h^2}{8}q_3\right) &= 0 \end{aligned} \qquad (6)$$

where

$$q_i = q_i^+(1+h/2R_1)(1+h/2R_2) + q_i^-(1-h/2R_1)(1-h/2R_2)$$
$$m_i = (h/2)[q_i^+(1+h/2R_1)(1+h/2R_2) - q_i^-(1-h/2R_1)(1-h/2R_2)] \quad (i = 1, 3)$$

Table 1 Influence of Thermal Expansion Through the Thickness on Central Lateral Deflection and Maximum Stress[a]

		Free-free						Fixed-fixed					
		$\alpha_{33} \neq 0$			$\alpha_{33} = 0$			$\alpha_{33} \neq 0$			$\alpha_{33} = 0$		
R (in.)	h (in.)	w_c (10^{-2} in.)	Q_{max} (lb/in.)	M_{max} (in.-lb/in.)	w_c	Q_{max}	M_{max}	w_c	Q_{max}	M_{max}	w_c	Q_{max}	M_{max}
	1.0	0.583	68.3	278	0.605	42.9	174	0.447	596	750	0.490	374	589
10	0.75	0.576	40.9	147	0.600	28.3	102	0.456	377	417	0.491	257.6	349
	0.50	0.575	20.4	58.4	0.596	15.8	45	0.458	205	187.7	0.484	150	165.6
	0.25	0.586	6.6	12.6	0.598	5.7	11	0.459	71	47.9	0.474	57.3	44.7
20		1.25	42.0	198	1.27	32.4	153	0.855	416	761	0.926	309	678
	1.0												
40		2.61	21.7	105	2.63	19.0	92	1.12	301	752	1.23	250	711

[a] $L = 15$, $E/G_c = 20$, w_c = lateral deflection at $L/2$.

The superscripts "+" and "−" identify surface forces at the outer ($\zeta = +h/2$) and inner ($\zeta = -h/2$) surfaces of the shell, respectively. In addition, we have the following natural boundary conditions at the α_1 edges:

$$\begin{aligned} \text{either } N_1 &= \overline{N_1} \quad \text{or} \quad u \text{ specified, and} \\ Q_1 &= \overline{Q_1} \quad \text{or} \quad w \text{ specified, and} \\ M_1 &= \overline{M_1} \quad \text{or} \quad \beta \text{ specified, and} \\ S_1 &= \overline{S_1} \quad \text{or} \quad \lambda \text{ specified, and} \\ T_1 &= \overline{T_1} \quad \text{or} \quad \gamma \text{ specified, and} \end{aligned} \tag{7}$$

Example Problem

The governing equations were solved using the multi-segment direct integration method developed by Kalnins[4] for solving linear and nonlinear two-point boundary value problems. The problem selected is that of a single-layer cylindrical shell ($A_1 = 1$, $R_1 = \infty$, $A_2 = R_2 = R$) of pyrolytic graphic subjected to a thermal gradient through the thickness. The temperature distribution was taken to be of the form $T(\zeta) = T_0 + (\zeta/h)T_1$ with $T_0 = T_1 = 1000°\text{F}$. Material properties were $E = E_1 = E_2 = 4.3 \times 10^6$ psi, $E_3 = 1.29 \times 10^6$ psi, $G_{13} = 2.15 \times 10^6$ psi, $\nu = \nu_{12} = \nu_{21} = -0.21$, $\nu_{23} = \nu_{13} = +0.90$, $\alpha = \alpha_{11} = \alpha_{22} = 0.60 \times 10^{-6}$ in./in.-°F, and $\alpha_{33} = 13.1 \times 10^{-6}$ in./in.-°F. The effects of E/G_{13} (in-plane Young's modulus), h/R (thickness to radius ratio), thermal expansion through the thickness and normal stress on stress resultants and lateral deflection were studied for both free-free and fixed-fixed boundary conditions. The computations were performed on a UNIVAC 1108.

Variation of shear stress and bending moment resultants with varying E/G_{13} for a free shell is shown in Fig. 1. Similar behavior is exhibited for a clamped shell. Even for shells considered geometrically thin ($h/R \leq 1/10$), significant errors can be induced by neglect of transverse shear deformation. Even greater errors are induced if thermal expansion through the thickness is neglected ($\alpha_{33} = 0$) as shown in Table 1 and Fig. 2. Thus, for the analysis of thin shells of anisotropic refractory materials and composites exhibiting properties similar to PG, both transverse shear deformation and thermal expansion through the thickness must be retained to have any hope of correct stress predictions. Use of classical shell theory wherein $e_{13} = \alpha_{33} = 0$ would lead to serious error.

To investigate the effects of normal stress due to mechanical coupling (as opposed to normal thermal stress which is accounted for by retention of α_{33}), computations were made using the present theory and one wherein $S_1 = T_1 = A = B = P_1 = P_2 = \lambda = \gamma = 0$ implying stress-strain relations of the form

$$\begin{aligned} e_{11} &= (\sigma_{11} - \nu\sigma_{22})/E + \alpha_{11}T \\ e_{22} &= (\sigma_{22} - \nu\sigma_{11})/E + \alpha_{11}T \\ e_{33} &= \alpha_{33}T \\ e_{13} &= \sigma_{13}/2G_{13} \end{aligned} \tag{8}$$

It is evident from Table 1 and Fig. 2 that for the unrestrained shell, transverse normal strain is produced by transverse thermal

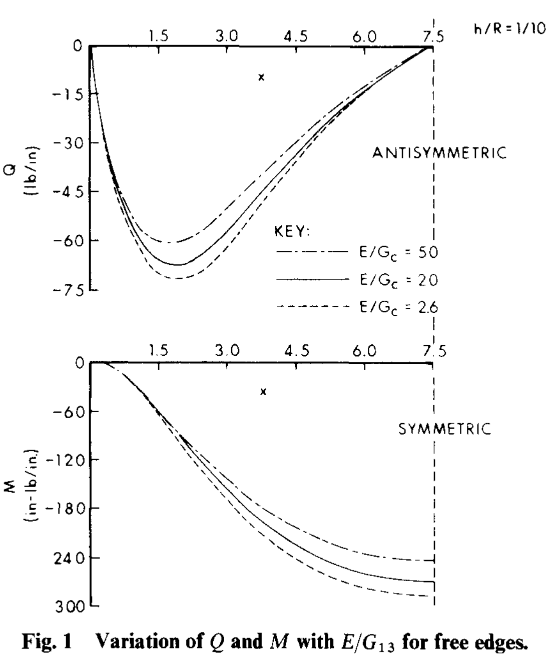

Fig. 1 Variation of Q and M with E/G_{13} for free edges.

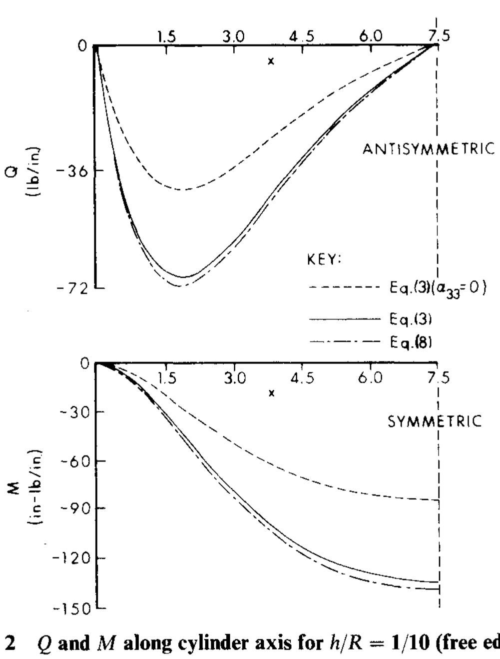

Fig. 2 Q and M along cylinder axis for $h/R = 1/10$ (free edges).

expansion and will not be affected by elastic strains. However, for the case of the clamped shell, neglect of transverse normal elastic strain will cause significant error in computing edge stresses. Thus, it would appear that for shells made of materials having high transverse Poisson's ratio (v_{13}), high thermal expansion coefficient ratios (α_{33}/α_{11}) and subjected to edge restraints, normal elastic, and thermal stresses are significant and must be accounted for in stress computations.

For those cases when elastic normal stress can be ignored in comparison with transverse normal thermal stress, a closed-form solution for both single and laminated shells[5] is possible for the static case.

References

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Mode Shapes and Frequencies of Clamped-Clamped Cylindrical Shells

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Nomenclature

a	= mean radius of shell
E	= modulus of elasticity
h	= thickness of shell wall
k	= $h^2/12a^2$
L	= length of shell
m	= number of axial half-waves
n	= number of circumferential waves
t	= time
u, v, w	= middle surface axial, circumferential, and radial displacements
x, ϕ	= axial and circumferential coordinates of shell middle surface
ν	= Poisson's ratio
ρ	= mass density of shell material
ω	= circular frequency
Ω	= frequency parameter, $\omega a[\rho(1-\nu^2)/E]^{1/2}$

Introduction

In a recent investigation into the influence of a blast environment on the dynamic response of a missile structure it became appropriate to examine the vibration characteristics of a finite length, thin cylindrical shell that was rigidly clamped at both ends. Although the shell vibration problem had previously been studied by several investigators, including Forsberg,[1,2] Smith and Haft,[3] and Vronay and Smith,[4] it became important in our case to develop a simple method for determining the shell's modal response to a uniform radial blast load. In the subsequent analysis it was noted that a misconception probably has arisen in previous comparisons between axisymmetric mode shapes with $n = 0$ and mode shapes with $n \geqq 1$.

An exact solution to Flügge's[5] thin shell equations was obtained using the approach suggested in Ref. 4, which eliminates the usual difficulty of deriving a new set of real solutions each time there is a change in the form of the roots of the shell's equations of motion.[1-3,6] This difficulty is overcome simply by working with an unaltered set of complex modal functions throughout the solution process. The process is carried out one step further in that the final set of mode shapes is taken directly from the real component of these complex modal functions.

Natural frequencies and mode shapes are presented here for the first six circumferential wave numbers ($n = 0$–5) and the two lowest radial modes of the shell model studied in Refs. 3 and 4. The special case of axisymmetric vibration ($n = 0$) is treated separately, but the case of beam-type vibrations ($n = 1$) is treated along with the higher wave numbers.

Solution of Shell Equations

In terms of the component displacements u, v, and w the general modal solution to Flügge's[5] differential equations of motion for a finite length, thin, circular, cylindrical shell can be written as

$$u = \sum_i A_i e^{\lambda_i x/a} \cos n\phi \sin \omega t$$
$$v = \sum_i B_i e^{\lambda_i x/a} \sin n\phi \sin \omega t \tag{1}$$
$$w = \sum_i C_i e^{\lambda_i x/a} \cos n\phi \sin \omega t$$

Using these mode shapes, the differential equations lead to two types of polynomial expansions in λ_i. For the axisymmetric case with $n = 0$ we obtain the sixth-order equation†

$$\lambda_i^6 + D_3\lambda_i^4 + D_2\lambda_i^2 + D_1 = 0 \tag{2}$$

but for all other cases, we get the eighth-order equation

$$\lambda_i^8 + D_4\lambda_i^6 + D_3\lambda_i^4 + D_2\lambda_i^2 + D_1 = 0 \tag{3}$$

where the coefficients D are functions of n, Ω, ν, and k and each λ_i is one of either a real or complex pair of roots.

The six roots of Eq. (2) ($i = 1$–6) and the eight roots of Eq. (3) ($i = 1$–8) can be used to solve for the constants A_i and B_i in terms of the C_i's by applying the homogeneous restrictions of the equations of motion (for the axisymmetric case $v = B_i = 0$). Typically each coefficient relationship will either be real or complex, so that the component modal displacements in Eq. (1) will be expressed, as in Ref. 4, as a sum of complex quantities.

We assume that both ends of the shell are clamped and axially constrained so that the eight boundary conditions are

$$u = v = w = \partial w/\partial x = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{4}$$

For the axisymmetric case, v is identically zero for all values of x, so that only six of these boundary conditions need be satisfied for $n = 0$.

Application of Eq. (1) to the boundary conditions leads to a set of homogeneous equations in terms of the unknown constants C_i

$$[Z]\{C_i\} = \{0\} \tag{5}$$

where $i = 1$–6 for $n = 0$; and $i = 1$–8 for $n \geqq 1$. The elements of the Z matrix are principally related to the frequency parameter Ω through either Eqs. (2) or (3). Since Eq. (5) is homogeneous, the remaining problem is to calculate the values of Ω that eventually will make the determinant $|Z|$ vanish.

The correct values Ω_j are essentially eigenvalues that determine the natural frequencies of the shell. The eigenvector constants $\{C_i\}_j$ corresponding to each Ω_j determine the shell's orthogonal mode shapes through Eq. (1). Whichever method is

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† For $n = 0$, a separate second-order solution also exists that involves only pure torsional motion.[2]